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Open Distance Pattern Coloring of Certain Classes of Graphs

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Abstract—Let G be a connected graph with diameter $d(G)$, $X=\{1,2,3,...,d(G)\}$ be a non-empty set of colors of cardinality $d(G)$, and let $\phi \neq M \subseteq V(G)$. Let $f_M{}^{o}$ be an assignment of subsets of X to the vertices of G such that $f_M{}^{o}(u)=\{d(u,v), v\in M, u\neq v\}$, where d(u,v) is the distance between u and v. We call $f_M{}^{o}$ an M-open distance pattern coloring of G if no two adjacent vertices have same $f_M{}^{o}$ and if such an M exists for a graph G, then G is called an open distance pattern colorable (odpc) graph; the minimum cardinality of such an M if it exists, is the open distance pattern coloring number of G denoted by $\eta_M(G)$. .In this paper, we study open distance pattern coloring of certain classes of graphs.

Index Terms— distance pattern coloring, open distance pattern of vertices, coloring, bipartite graphs,chain graphs, triangular snake, quadrilateral snake,

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1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [10] and for more about graph labeling, we refer to [12]. Unless mentioned otherwise, all graphs considered here are simple, finite and connected. Let G be a (p,q)-graph and let X, Y and Z be non-empty sets and $2^X, 2^Y$ and 2^Z be their power sets. Then, the functions $f: V(G)\rightarrow 2^X$, and $f: E(G)\rightarrow 2^Y$ $f: V(G)\cup E(G)\rightarrow 2^Z$ are called the *set assignments* of vertices, edges and elements of G respectively. By a set-assignment of a graph, we mean any one of them. A set-assignment $f: V(G)\rightarrow 2^X$ is called a *set-labeling* or a *set-valuation* if it is injective. A *proper coloring* of a graph G is a function from the vertices of G to a set of colors such that no two adjacent vertices have the same color. The *chromatic number* of a graph G is the minimum number of colors required in its proper coloring. Graph coloring has been used as a model in many practical problems and has played a vital role in the development of graph theory. Using the concepts of graph coloring, distances in graphs and set-labeling of graphs, we defined the following in [7].

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22.Definition.1.1 [6] Given a connected graph G(V,E) of diameter $d(G)$,$\phi \neq M \subseteq V(G)$. Let , $X=\{1,2,3,...,d(G)\}$ be nonempty set of colors of G with cardinality d(G). Let be $f_M{}^{o}$ an assignment of subsets of X to the vertices of G such that $f_M{}^{o}(u)=\{d(u,v), v\in M, u\neq v\}$, where d(u,v) is the usual distance between u and v. We call $f_M{}^{o}$ an M-open distance pattern coloring of G, if no two adjacent vertices have same $f_M{}^{o}$ and if such an M exists for a graph G, then G is called an *open distance pattern colorable graph*. An open distance pattern colorable graph is usually written in short as an *odpc-graph.* The minimum cardinality of such a set M, if it exists, is said to the *open distance pattern coloring number* (odpc-number, in short) of G, denoted by $\eta_M(G)$.
It has been proved, in [6], that for any graph G $\eta_M(G)\geq 2$. .Further, the following theorem has been proved in [6].

Theorem 1.2. [6] *Every connected bipartite graphs are open distance pattern colorable*.
In this paper, we study open distance pattern coloring of certain classes of graphs.
2 **Main Results**:

The graph obtained by identifying the end points of *b* internally disjoint paths, each of length *a*, is denoted, in [4], by $P_{a,b}$. The following proposition establishes the open distance pattern colorability of this graph class.

Proposition 2.1 $P_{a,b}$ *is open distance pattern colorable*.

Proof. Let the end points of b internally disjoint paths of length a are identified at u and *v*, Hence, any cycle in the graph $P_{a,b}$ is of length 2a. That is, the length of any cycle in $P_{a,b}$ is even and hence it is a bipartite graph. Therefore, by Theorem 1.2, is o $P_{a,b}$ **o**pen distance pattern colorable.

Theorem 2.2. *The graph G isomorphic to n cycles C_m all of which have one edge in common is odpc if and only if* $m \geq 4$

..Proof. Let $V(G) = \{u_{ij}, 1 \leq i \leq n, 1 \leq j \leq m\}$

be the vertex set of *G*, where u_{ij} is the vertex set of the ith-copy of C_m
Assume that $m \geq 4$. Then, we have the following cases.
Case 1: m even. *G* is bipartite. By theorem 1.2, G is odpc.
Case 2: m odd. Choose $M = \{u, u_{r\lceil m/2 \rceil}, u_{s\lceil m/2 \rceil}\}$. Then $f_M{}^o(u) = \{\lfloor m/2 \rfloor\}$, $f_M{}^o(v) = \{1, \lfloor m/2 \rfloor\}$. Also $f_M{}^o(u_{r\lceil m/2 \rceil}) = f_M{}^o(u_{s\lceil m/2 \rceil}) = \{m-1, \lfloor m/2 \rfloor\}$.

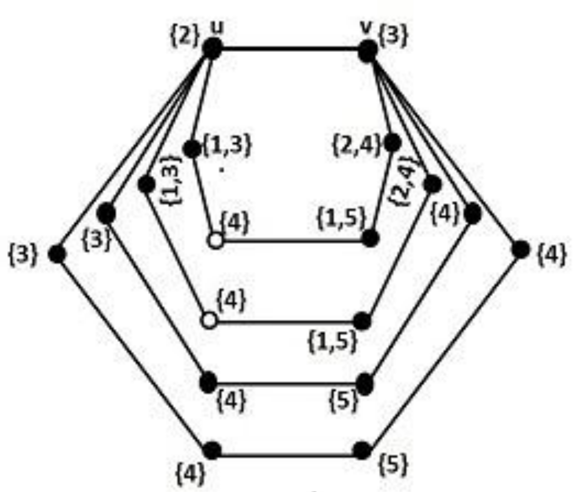
For $i = r, s$; $f_M{}^o(u_{ij}) = \{j-1, \lceil m/2 \rceil + j - 2, \lceil m/2 \rceil - j\}$ for j =2,3,4,…,$\lceil m/2 \rceil$-1 and ,

$f_M{}^o(u_{ij}) = \{m-(j-1), j - \lceil m/2 \rceil, \lceil m/2 \rceil + m - (j+1)\}$ for $j = \lceil m/2 \rceil + 1, ..., m-1$.

For $i \neq r$ and $i \neq s$, $f_M{}^o(u_{ij}) = \{j-1, \lceil m/2 \rceil + j - 2\}$ for j={2,3,…,$\lceil m/2 \rceil$} and

$f_M{}^o(u_{ij}) = \{m - j + 1, \lceil m/2 \rceil + m - j - 1\}$ for $j = \lceil m/2 \rceil + 1, ..., m-1$.

From all the above cases, it is evident that no two adjacent vertices of G have the same $f_M{}^o$. Hence, *G* is open distance pattern colorable.
Conversely, assume that *G* is open distance pattern colorable. If possible, let m=3
Then $G \cong K_2 + K_n$. Let $V(K_2) = \{v_1, v_2\}$. $V(Kn) = \{v_3, v_4, ..., v_n, v_{n+1}, v_{n+2}\}$
. $V(K_n) = \{v_3, v_4, ..., v_{n+1}, v_{n+2}\}$.
If we choose any number of vertices of *G* to *M*, $f_M{}^o(v_1) = f_M{}^o(v_2) = \{1\}$.
Therefore, *G* is not open distance pattern colorable. This completes the proof.
Figure 1 depicts odpc labeling of *4* copies of C_6 which have one edge in common. The vertices in M are represented by white circles in the figure.



Another interesting graph structure is the path union of a given graph G, which is defined as follows.

Definition 2.3. [12] Let *G1,G2,G3 ,…,Gn* be *n* copies of a given graph *G*. The graph obtained by adding an edge from Gi to Gi+1 for all i=1,2,3,…,n-1 is called the path union of *G*.
We now proceed to verify the open distance pattern colorability of the path union of *G* in the following theorem.

Theorem 2.4. *Let G be the path union of m copies* ($m \geq 2$) *of cycle Cn. Then, G is open pattern distance colorable except when m is even and n = 3.*

Proof. Let G be the path-union of *m* copies of the cycle C_n .Consider the following cases.

Case 1: n even. Then, *G* can be considered as the union of even cycles and hence is a bipartite graph. Therefore, by Theorem 1.2, G is open distance pattern colorable.
Case 2: n odd. Here we have the following subcases.
Subcase 2.1: m odd.
Choose the set $M = \{v_{12}, v_{22}, ..., v_{m2}\}$. For $i = 1,2,3,..., \lfloor m/2 \rfloor$,
$f_M{}^o(v_{i1}) = f_M{}^o(v_{m-(i-1)1}) = \{1,2,3,..., m-(i-1)\}$,
$f_M{}^o(v_{\lceil m/2 \rceil 1}) = \{1,2,3,..., \lceil m/2 \rceil\}$.
For the vertex v_{i2}, there are two adjacent vertices at distance diameter of the cycle. These two vertices have identical element $\lceil m/2 \rceil$ in their $f_M{}^o$. By considering distance from these two vertices to other elements in *M*, they differ by 1. Hence adjacent vertices have distinct . $f_M{}^o$.

Subcase 2.2: *m* is even and *n = 3*. Assume that *n = 3*. No vertices of the form v_{i1} can not be an element of *M*, since if it is so, the vertices v_{i2}, v_{i3} have $f_M{}^o(v_{i2}) = f_M{}^o(v_{i3})$ for any *i*. If we take any number of vertices of the form v_{ij}, $j \neq 1$, then the vertices $v_{\frac{m}{2}1}$ and $v_{((\frac{m}{2})+1)1}$ have the same distance pattern. Therefore, *G* is not open distance pattern colorable if *n = 3*.
Subcase 2.3: *m* is even and n ≥ 5. In this case, choose $M = \{v_{11}, v_{1\lceil n/2 \rceil}, v_{22}, v_{31}, v_{3\lceil n/2 \rceil}, ..., v_{(m-1)1}, v_{(m-1)\lceil n/2 \rceil}, v_{m2}\}$
Then, for $i = 1,2,3,..., m$
(a) If i is odd, then 2 is an element of $f_M{}^o(v_{i1})$, but 1 is not in $f_M{}^o(v_{i1})$. .Moreover, two vertices v_{ij} are equidistant from v_{i1} and distance of these vertices from other elements in M differ by 1. Hence adjacent vertices have distinct $f_M{}^o$.
(b) If i is even, then *1* is an element of $f_M{}^o(v_{i1})$,but *2* is not in $f_M{}^o(v_{i1})$ .Moreover, v_{ij} for $j = \lceil n/2 \rceil, \lceil n/2 \rceil + 1$ are equidistant from v_{i1} and distance of these vertices from other elements in M differ by 1. Hence adjacent vertices have distinct $f_M{}^o$. This completes the proof.

Definition 2.5. [12] A triangular snake, denoted by S_{3n} is the graph obtained from a path P_n by replacing every edge of it by a cycle C_3.
Theorem 2.6. *A triangular snake* S_{3n} *is open distance pattern colorable if* $n \neq 3$.
Proof. Let $P_n : u_0 u_1 u_2 ... u_n$. For $0 \leq i \leq n-1$ the triangular snake S_{3n} is the graph obtained by replacing every edge of P_n by the triangle $u_i u_{i+1} v_{i+1} u_i$. We prove the theorem in

three cases.

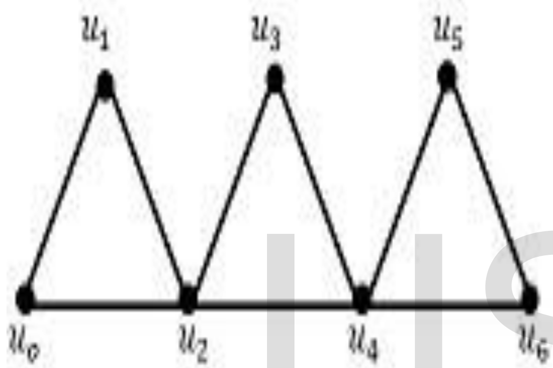
Case 1: When $n = 2$

Choose $M = \{u_0, u_2\}$. Then $f_M^o(u_1) = \{1\}$ $f_M{}^o(u_0) = f_M{}^o(u_2) = \{2\}$ and $f_M^o(v_1) = f_M^o(v_2) = \{1,2\}$.

Case :2 When $n \geq 4$. For the choice of $M = \{u_0, u_2, u_n\}$, we have $f_M^o(u_0) = \{2, d(G)\}$, $f_M^o(u_1) = \{1, d(G)-1\}$, and $f_M^o(v_1) = \{1, d(G)\}$, $f_M^o(v_2) = \{1,2, d(G)-1\}$, and for $3 \leq i \leq n$,

$f_M^o(u_i) = \{i, i-2, n-i\}, f_M^o(v_i) = \{i, i-2, n-(i-1)\}$. With this choice of M, $S_{3,n}, n \neq 3$ is odpc.

Case 3: If possible let $n = 3$, $S_{3,3}$ is odpc. Label the vertices as shown in Figure 2.



If neither u_0 nor u_1 is in M, then $f_M^o(u_0) = f_M^o(u_1)$. Hence, either u_0 or u_1 must be an element of M. By the same argument we see that either u_5 or u_6 belongs to M. Now, let u_0 and u_6 are in M. Then $f_M^o(u_2) = f_M^o(u_4) = \{1,2\}$ irrespective of the case whether u_2, u_3 are in M or not. Since u_2 is at distance 1 from u_0 and u_1, at distance 2 from u_5 and u_6, and u_4 is at a distance 1 from u_5 and u_6, and at distance 2 from u_0 and u_1 $f_M^o(u_2) = f_M^o(u_4) = \{1,2\}$ in all possible cases. Hence $S_{3,n}$ $n \neq 3$ is not odpc.

Analogous to triangular snake, a quadrilateral snake is defined as follows;

Definition 2.7. A quadrilateral snake, denoted by $S_{4,n}$ is the graph obtained from a path P_n by replacing every edge of it by a cycle C_4.

Theorem 2.8. *A quadrilateral snake is open distance pattern colorable.*

Proof. A quadrilateral snake is a graph that has only cycles of of length 4 and hence is bipartite. Therefore, by Theorem 1.2, G is open distance pattern colorable.

Another interesting graph we consider is a chain graph which is defined as follows.

Definition 2.9. [1] A chain graph is a graph with blocks $B_1, B_2, B_3, \dots, B_n$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block cut point is a path.

Definition 2.10. [15] A chain graph with n blocks and the sequence of n blocks of complete graphs $(K_{a1}), (K_{a2}), \dots, (K_{an})$ is called a Husimi Chain and is denoted by $CK(n; (a_1, a_2, a_3 \dots, a_n)); a_i \geq 2$.

If $a_1 = a_2 = a_3 = \cdots = a_n = 2$, then $CK(n; (2,2,2,\dots,2)) = P_n$, a path of length $n \geq 3$ and if $a_1 = a_2 = a_3 = \cdots = a_n = 3$, then $CK(n; (3,3,\dots,3)) = S_{3n}$ a triangular snake with $n \neq 3$. In both cases, G is odpc, by Theorem 1.2 and Theorem 2.6 respectively. It is meaningless to say that P_n has an open distance pattern colorable for $n \leq 2$ and we have already proved in Theorem 2.6 that $S_{3,n}$ not odpc if $n = 3$. It remains to verify the other cases.

Theorem 2.11. *$G = CK(n; (a_1, a_2, a_3 \dots, a_n))$ is not an odpc-graph if $a_i \geq 4$ for some i; $1 \leq i \leq n$.*

Proof. For some i; $1 \leq i \leq n$ assume that $a_i \geq 4$. Let $u_1, u_2, u_3 \dots, u_{ai}$ be the vertices of the component K_{ai}. We consider the following cases.

Case 1: If K_{ai} is an end component of G, then exactly one vertex of K_{ai} is common to another component $K_{a\,j}$ of G. Without loss of generality, let u_1 be the vertex of K_{ai} that is common to the component $K_{a\,j}$. Then, there are the following subcases.

Subcase 1.1: When $u_2, u_3 \dots, u_{ai}$ are not the elements of M.

In this case, for some positive integer k, let $\{i_1, i_2, i_3 \dots, i_k\}$ be the set assignment f_M^o of u_1 with respect to M. If $u_1 \notin M$, for all $2 \leq r \leq ai$, $f_M{}^o(u_r) = \{i_1+1, i_2+1, \dots, i_k+1\}$

That is , the adjacent vertices u_r have the same set assignment for all $2 \leq r \leq a_i$. Hence if there exists an odpc set for the graph G, then necessarily $u_1 \in M$. Then, for positive integer t, let $f_M^o(u_1)$ $=\{i_1, i_2, i_3 \dots, i_t\}$. Therefore $f_M^o(u_r)$ $=\{i_1+1, i_2+1, i_{3\}}+1, \dots, i_t+1\}$; $2 \leq r \leq ai$.

Subcase 1.2: When one of $u_2, u_3 \dots, u_{ai}$ is in M. In this case $u_1 \notin M$. Without loss of generality, let $u_2 \in M$. $f_M{}^o$ (u_2) $=\{i_1, i_2, i_3 \dots, i_s\}$ for some positive integer s. Then f_M^o (u_r) $=\{1, i_1, i_2, i_3 \dots, i_s\}$

Case2: If K_{ai} is internal component of G and let K_{a1}, K_{a2}, $K_{a3}, \dots, K_{ai}$ be the left Husimi chain and $K_{ai}, K_{ai+1}, \dots, K_{an}$ be the right Husimichain of $CK(n; ((a_1, a_2, a_3 \dots, a_n))$. We can adopt the process, same as in case 1, for both the chains.

For $1 \leq l \leq a_i$, let u_l be a vertex of K_{ai}, which is not common to any other component of G. If f_M^o (u_l) $=\{i_1, i_2, i_3 \dots, i_r\}$; $u_l \in K_{ai}$ with respect to odpc set M_1, in left Husimi chain and if f_M^o (u_l) $=\{j_1, j_2, j_3 \dots, j_s\}$; $u_l \in K_{ai}$ with respect to the odpc set M_2, in the right Husimi chain. Then, f_M^o (u_l) with respect to the set $M = M_1 \cup M_2 = \{i_1, i_2, i_3 \dots, i_r\} \cup \{j_1, j_2, j_3 \dots, j_s\}$. In all these cases M cannot be an odpc-set.

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